Symmetries in Gauge Theories and their Applications

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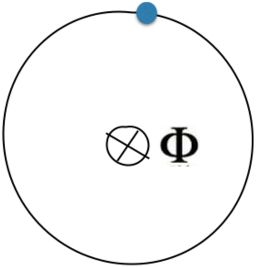
The most important open problems in many body physics are associated with strong coupling, for example,

- Planck scale physics
- Confinement in Yang Mills theory
- High T_c superconductivity
- Ising model and more general magnetism in $d = 3$
- Nontrivial spin liquid phases (e.g. Herbertsmithite)
- etc.

I would like to describe a technique, developed with **Gaiotto, Kapustin, Seiberg, ZK**, which allows to prove exact results about Yang-Mills theory and various other models.

I will start with an easy quantum mechanical model which serves as a pedagogical example and introduces the relevant physical and mathematical machinery.

Related, works are [A.Sharon, R.Thorngren, X.Zhou, ZK], [T.Sulejmanpasic, M.Unsal, ZK], [J.Gomis, N.Seiberg, ZK].


Φ

The classical Lagrangian of a particle on a ring is

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

and $V(q)$ being some potential such that $V(q + 2\pi) = V(q)$.

For example for $V = 0$ we could have the particle moving around with arbitrary angular velocity $q = \omega t$, and energy $E = \frac{1}{2}\omega^2$.

The first surprise about this model is that the quantization is ambiguous. There is a one parameter family of choices because the wave function does not need to be periodic

$$\Psi(q+2\pi)=e^{i\theta}\Psi(q)$$

Physically, this is due to a solenoid with $\Phi=\theta$ magnetic flux

Equivalently, in terms of Feynman's sum over histories, we can break up the space of possible histories according to how many times one winds around the circle and add an appropriate phase. So the Lagrangian is really

$$L = \frac{1}{2}\dot{q}^2 - V(q) + \frac{\theta}{2\pi}\dot{q}$$

and now the wave functions are taken to be always periodic.

The extra term in the path integral,

$$Exp\left[i\frac{\theta}{2\pi}\int dt\dot{q}\right]$$

is just

$$Exp\left[i\theta n\right]$$

where n is the number of time the particular path winds around the circle.

Note that the deformation parameter θ is periodic. This is manifest in all the descriptions we have given so far

$$\theta \simeq \theta + 2\pi \ .$$

Let us take $V = 0$.
The conjugate momentum is shifted as $\Pi_q = \dot{q} + \frac{\theta}{2\pi}$ and the Hamiltonian is thus

$$H = \frac{1}{2}\left(\Pi_q - \frac{\theta}{2\pi}\right)^2$$

The eigenfunctions and eigenvalues are

$$|n\rangle \ , \qquad \langle q|n\rangle = \Psi_n(q) = e^{iqn} \ .$$

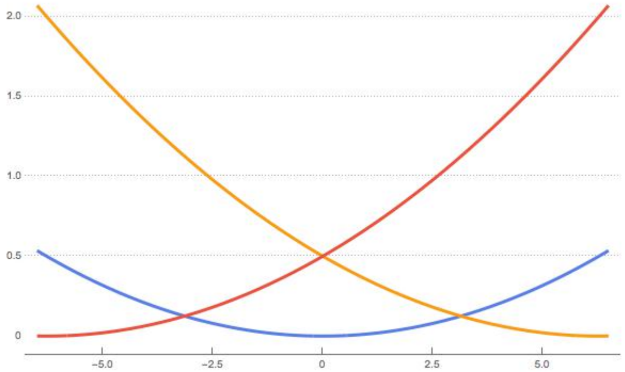
$$E_n = \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2 \ .$$

We see that the full spectrum is indeed invariant under $\theta \to \theta + 2\pi$. But the individual states cross each other so we have a nontrivial map where

$$\theta \to \theta + 2\pi$$

is accompanied by

$$|n\rangle \to |n+1\rangle \ .$$


2.0
1.5
1.0
0.5
0
-5.0
-2.5
0
2.5
5.0

One should say that the theories with θ and $\theta + 2\pi$ are equivalent because there is a similarity transformation between them. The similarity transformation is however nontrivial. Indeed, it is implemented by the operator

$$U = e^{iq} \ , \qquad U^\dagger U = 1 \ .$$

A curious case is $\theta = \pi$ where we see that the ground state is doubly degenerate – it is a qubit. For example at $\theta = \pi$ the ground states are spanned by

$$|0\rangle, |1\rangle \ .$$

So far this is a free theory. But now let us discuss what happens if we add a potential

$$V(q) = \sum_{k\in\mathbb{Z}} c_k \cos(kq) + d_k \sin(kq)$$

I will introduce some tools that would allow to prove the following theorem:

If only $c_{2k} \neq 0$ then the qubit at $\theta = \pi$ remains.

So for example we expect that $V(q) = cos(q)$ would lift the qubit but $V(q) = cos(2q)$ would retain it.

This is strange!
$V(q) = cos(2q)$ has two ground states ($q = \pi/2, 3\pi/2$) and we are used to instantons lifting classical vacuum degeneracy in quantum mechanics.

We begin by considering again the free model $V = 0$. Obviously it has a rotation symmetry $q \to q + \alpha$ for any α. This symmetry group is

$$SO(2)$$

At $\theta = 0, \pi$ there is another symmetry $q \to -q$.

Note that to see that $q \to -q$ is a symmetry at $\theta = \pi$ we used the fact that $\theta = \pi$, and $\theta = -\pi$ describe the same theory. So the symmetry group at $\theta = 0, \pi$ is

$$O(2)$$

The generator of the reflection is denoted by C and the generator of $SO(2)$ is V_α. The group $O(2)$ means that we have

$$CV_\alpha C = V_{-\alpha} \ , \qquad C^2 = 1 \ , \qquad \alpha \simeq \alpha + 2\pi \ .$$

Now we should realize them on the Hilbert space:

$\underline{\theta = 0:}$ $V_\alpha |n\rangle = e^{in\alpha}|n\rangle, \ C|n\rangle = |-n\rangle.$

$\underline{\theta = \pi:}$ $V_\alpha |n\rangle = e^{in\alpha}|n\rangle, \ C|n\rangle = |-n+1\rangle.$

$\underline{\theta = 0:}$ $V_\alpha|n\rangle = e^{in\alpha}|n\rangle$, $C|n\rangle = |-n\rangle$.
This is completely consistent with $C^2 = 1$, $\alpha \simeq \alpha + 2\pi$ and $CV_\alpha C = V_{-\alpha}$. The ground state $|0\rangle$ is unique, and $O(2)$ invariant.

$\underline{\theta = \pi:}$ $V_\alpha|n\rangle = e^{in\alpha}|n\rangle$, $C|n\rangle = |-n+1\rangle$.
$C^2 = 1$, $\alpha \simeq \alpha + 2\pi$ are obeyed. But if we try to check the algebra of charges we find that

$$CV_\alpha C = e^{i\alpha} V_{-\alpha}$$

We see that the algebra of operators is centrally extended when we act on the Hilbert space!
So the operator algebra is $O(2)$ but the representation on the Hilbert space is a central extension, called $Pin(2)$.

Indeed, we cannot remove this central extension, and it is $\mathbb{Z}_2$ valued.

$$CV_\alpha C = e^{i\alpha} V_{-\alpha}$$

define $U_\alpha = V_\alpha e^{-i\alpha/2}$, then

$$CU_\alpha C = U_{-\alpha} \ .$$

But now $\alpha \ \simeq \alpha + 4\pi$ since $U_{2\pi} = -1$. This is very similar to how $SU(2)$ is the double cover of $SO(3)$. Here we have

$$\mathbb{Z}_2 \to Pin(2) \to O(2)$$

A neat way to summarize this curious situation is to say that the classical symmetry $O(2)$ suffers from a 't Hooft anomaly in the quantum theory. In particular, the ground state is not a representation of $O(2)$ but of $Pin(2)$. It is therefore natural to give the $|0\rangle, |1\rangle$ states charge $1/2$ under the rotations. So the particle $q(t)$ is really a **fermion** at $\theta = \pi$.

Indeed, the ground state of the model is dual to the free fermion

$$\mathcal{L} = \psi^\dagger \dot{\psi} \ .$$

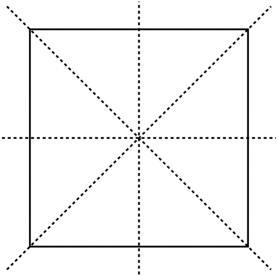
We are thus dealing here with a system that has a $\mathbb{Z}_2$-valued anomaly for its $O(2)$ global symmetry. But do we really need to preserve the full $O(2)$ for that? Suppose we only retain V_π and C. Classically they just generate $\mathbb{Z}_2 \times \mathbb{Z}_2$. Quantum mechanically we have a central extension η.

$$V_\pi^2 = C^2 = 1 \ , \qquad CV_\pi C = \eta V_\pi \ , \qquad \eta^2 = 1 \ ,$$

and $\eta = -1$ at $\theta = \pi$ and $\eta = 1$ for $\theta = 0$. η is central.

The group generated by V_π, C, η is just the dihedral group that preserves the square, D_8.
So while the symmetry at $\theta = 0$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, at $\theta = \pi$ it is centrally extended to D_8.

The ground state qubit $|0\rangle \oplus |1\rangle$ furnishes the two dimensional representation of D_8 where $|0\rangle$ and $|1\rangle$ are like the sides of a square.



Since our $\mathbb{Z}_2$-valued anomaly exists even if we break $O(2)$ down to $\mathbb{Z}_2 \times \mathbb{Z}_2$, we can now add any potential which has this symmetry. Since the anomaly cannot disappear the degeneracy has to remain! For example, any potential of the form

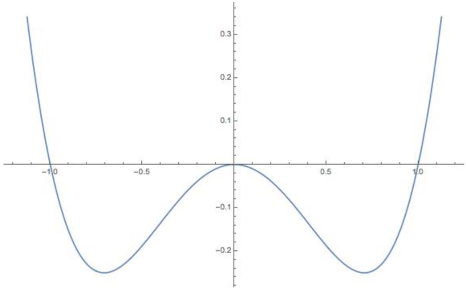
$$V(q) = \sum_{k \in \mathbb{Z}} c_{2k} \cos(2kq)$$

would work since it preserves $q \to -q$ and $q \to q + \pi$.

The degeneracy between the two classical minimal of $cos(2q)$ is preserved in the exact quantum theory. As we said already, the folklore is that instantons always lift the degeneracy in quantum mechanics and the ground state is unique (at least in non-supersymmetric cases). We see that this is not true.

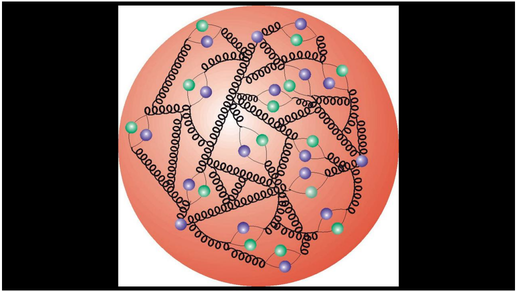
The point is that all the instantons cancel each other exactly.

In a potential like the double well the instantons do not cancel.



But in our case they do.

This example that we discussed turns out to be very educational because almost all the aspects that we discussed carry over to Yang-Mills theory!



First let us review the classical Yang-Mills theory with $SU(N)$ gauge group. The dynamical variable, a, is a space-time vector which is also traceless Hermitian $N \times N$ matrix and we define

$$F = da + i[a, a] \ .$$

We write the classical action/energy functional

$$S = \int d^4x Tr(F \wedge \star F) = \int d^4x |F|^2 \ .$$

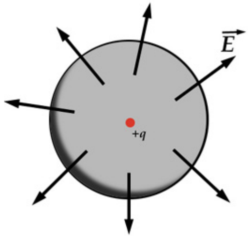
The Euler Lagrangian equations, known as the Yang-Mills equations, are very nonlinear

$$d_a F = d_a \star F = 0 \ ,$$

where $\star$ is the Hodge dual.

There is of course a huge amount of work on these classical PDEs.

This theory is a nonlinear version of Maxwell's theory. The crucial physical difference is that the fields E, B in Maxwell's theory do not carry electric charge. This is why it is linear. Here E and B carry charge. This leads to an important difference. In Maxwell's theory we have a Gauss Law



we can extract the total charge by measuring

$$\int_{S^2} E$$

But in Yang-Mills theory we cannot quite do that because the field itself is charged so it is hard to isolate the probe charge. So imagine that we put an external probe particle in the representation R of $SU(N)$. The fields E, B are in the representation $Adj(SU(N))$ and hence they may confuse us and we can measure any charge of the form

$$R \otimes Adj \otimes Adj \cdots \ .$$

So we cannot measure R but we can measure it N-ality. We thus have a $\mathbb{Z}_N$ "center" symmetry. In Maxwell's theory it is $U(1)$ because we can measure any charge.

For example, in $SU(2)$ Yang-Mills, we can measure whether R is even- or odd- dimensional representation but not more than that.

Following [Gaiotto, Kapustin, Seiberg, Willett] this center symmetry is called "one-form symmetry."
It is essentially a discrete version of the Gauss law, as we explained above.

The quantum theory is, however, non-unique! The physical reason is as before – in the various possible histories, there is a hidden internal circle and the histories can be divided according to how many times this circle has been wound around.

$$\frac{1}{2\pi}\int dx\ \dot{q} \longleftrightarrow \frac{1}{8\pi^2}\int d^4x\ Tr(F\wedge F)$$

Therefore in the quantum theory we have as before a θ parameter which we can add to the action

$$\frac{i\theta}{8\pi^2}\int d^4x Tr(F\wedge F)$$

and it is manifest that $\theta \simeq \theta + 2\pi$.

The two choices $\theta = 0, \pi$ are special! They preserve time-reversal symmetry. Or equivalently, they preserve CP symmetry. This is again slightly nontrivial at $\theta = \pi$ since we need to use the fact that $\theta = \pi$ and $\theta = -\pi$ describe the exact same theory. But there is, as before, a nontrivial similarity transformation involved at $\theta = \pi$. The operator that implements the similarity transformation is the Chern-Simons action $U = e^{\frac{i}{4\pi}\int\left(ada+\frac{2}{3}a^3\right)}$ – if we conjugate the Hamiltonian by it, then $\theta \to \theta + 2\pi$.

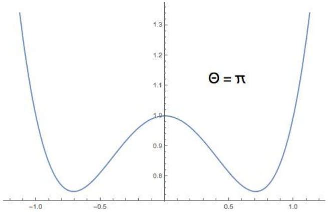
Now we need to discuss the analog of the $SO(2)$ symmetry, $q \to q + \alpha$. This role is played by the center symmetry. In Maxwell's theory it is $U(1)$ and in $SU(N)$ Yang-Mills theory it is $\mathbb{Z}_N$.

The symmetry group at $\theta = 0, \pi$ is therefore

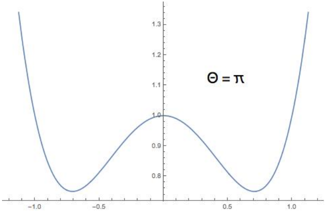
$$\mathbb{Z}_2^T \times \mathbb{Z}_N$$

At $\theta = 0$ there is no anomaly and for $\theta = \pi$ there is an anomaly.

The model at $\theta = 0$ is gapped with a trivial confining vacuum, as observed by lattice simulations. The model at $\theta = \pi$ however cannot have a trivial ground state. One way to saturate the anomaly is to **break time reversal, $\mathbb{Z}_2^T$, spontaneously** and have two degenerate confining ground states.



What happens if we try to interpolate between these two vacua?



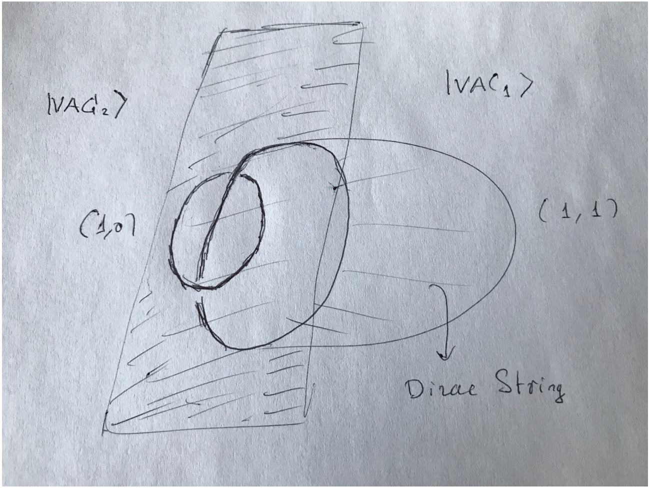
We get a domain wall, where on one side we have the first vacuum and on the other side we have the second vacuum. The domain wall cannot be trivial – it must carry a

$$SU(N) \text{ level 1 Chern} - \text{Simons Theory}$$

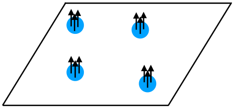
$$S = \frac{1}{4\pi}\int_3 Tr\left(ada + \frac{2}{3}a^3\right)$$

While the bulk is gapped, this prediction means that there is topological order on the interface. The anyons are $Pe^{i\int a}$, with a taken in some representation R. These anyons are deconfined on the interface.

To summarize, unlike the quarks in the bulk, the quarks are deconfined on the wall and furthermore the quarks become anyonic even though their original spin is integer or half integer. For example, the spin of the fundamental quark is now $\frac{1}{2N} + \frac{1}{2} \mod 1$.



This is the starting point of an intriguing connection between the dynamics of four-dimensional theories and three-dimensional theories. One can actually derive many 3d dualities by following this strategy of studying anomalies and domain walls in four-dimensional gauge theories.

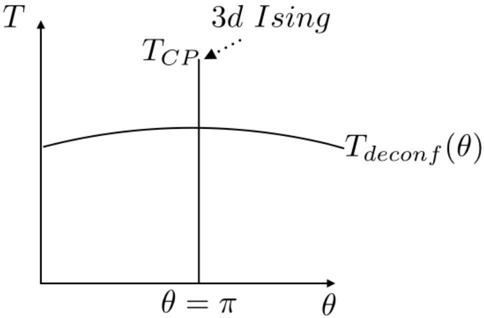


Finite Temperature?

As we crank up the temperature we expect time reversal, $\mathbb{Z}_2^T$, would be eventually restored. Also, we expect the gluons would be liberated (deconfined) in the bulk. The anomaly can be saturated only if the latter happens before the former.

Therefore, at sufficiently large N (probably $N \geq 3$) we expect the following phase diagram for Yang-Mills theory

$$T_{CP} \geq T_{deconf}(\theta = \pi)$$



I am not sure what is the phase diagram for $SU(2)$ Yang-Mills theory; there are several options which are logically possible and compatible with the anomaly. We will have to wait for simulations to settle this debate.

Let us now briefly discuss the question of whether this mechanism operates in nature. First of all, in QCD there are fundamental quarks and the $\mathbb{Z}_N$ is thus not present. But there is a symmetry that rotates between the quarks which in some sense replaces it. If one of the quark masses is negative, e.g. if $m_u < 0$ then one may have spontaneously broken time reversal invariance very similar to what we described.

There could be an additional sector to the Standard Model where the above mechanism occurs. Perhaps this could trigger the violation of CP that we see in the real world.

We saw that using anomalies and topology one can establish non-perturbative results about Yang-Mills theory. But there are many other examples where we can make striking predictions using new discrete 't Hooft anomalies, for example,

- QCD and many generalizations.
- Néel-VBS transitions in quantum anti-ferromagnets and generalizations.
- Chern-Simons theories.
- Sigma models in two dimensions.
- etc.

Summary

- Topology and anomalies offer a window into non-perturbative physics, leading to concrete, verifiable predictions.
- The mathematical underpinnings of these developments lie in understanding discrete characteristic classes of principle bundles (and also higher gauge theory).
- There are applications for domain walls, thermal physics (with and without chemical potentials), the nature of various transitions (1st/2nd order), and dualities.