Chapter 1

Line Defects and the Renormalization Group

Zohar Komargodski
Simons Center for Geometry and Physics
Stony Brook, New York, USA
zkomargo@gmail.com

We define a notion of zero-temperature entropy for impurities (line defects) in $d+1$ space-time dimensions. We show that this entropy obeys a simple evolution equation under the renormalization group. We apply this result for impurities in magnets and in gauge theories. We find new critical impurities and phase transitions.

1. General Introduction

Conformal Field Theories (CFTs) in $d+1$ space-time dimensions are continuum theories describing second-order phase transitions. They arise in a variety of physical constructions, for instance

- Certain zero-temperature quantum systems in d space dimensions at loci of the coupling constants corresponding to a diverging correlation length. Here the relativistic invariance in $d+1$ space-time dimensions and the emerging light-cone hold only near the phase transition.
- Classical statistical systems in thermal equillibrium in $d+1$ Euclidean dimensions at a point with a diverging correlation length.
- Quantum systems in $d+1$ space dimensions where one space dimension is taken to be a circle whose radius is tuned to a point with a diverging correlation length.

There are many other instances in which CFTs are encountered.

The symmetry algebra of the continuum theory is $SO(d+1,2)$, which includes space-time rotations and boosts, translations, dilation (scaling) symmetry $x^\mu \to \lambda x^\mu$ (with $x^\mu \equiv (t, x^i)$) and special conformal transformations, which are generated by combining an inversion $x^\mu \to x^\mu/x^2$ with a translation in some direction b^μ followed by another inversion: $\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu$.

A central problem about CFTs is to understand the space of local operators $\mathcal{O}_i(x)$, which can be chosen to carry a well defined scaling dimension Δ_i and spin

J_i. The scaling dimensions have played a significant role in the history of the subject due to the relation between the scaling dimensions of the low-lying operators and the critical exponents near the phase transition.[1]

A less trodden path is the study of extended operators in CFTs. Relativistic invariance and the Wick rotation allow us to think of extended operators as either nonlocal operators acting at a given time, or as a modification of the Hamiltonian by an insertion of an impurity in some region of space. The simplest example is line operators, which can be viewed as either extended operators localized on a line acting on the usual Hilbert space of the theory or, equivalently, as a localized point-like impurity in space, say at $x^i = 0$. In the latter point of view, the impurity potentially changes the Hilbert space, the space of local operators acting on it, and a new ground state must be found in the presence of the impurity.

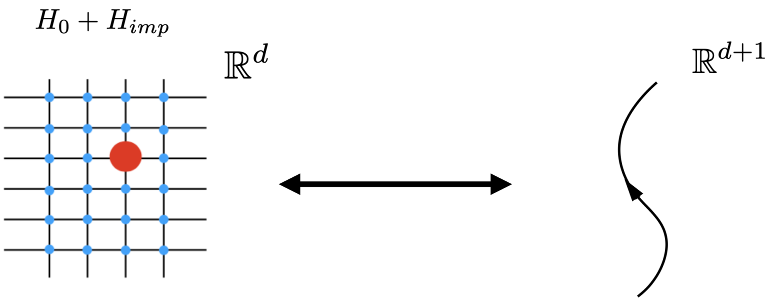

Fig. 1. Two ways of thinking about line defects: either as a modification of the Hamiltonian due to a point-like impurity in $\mathbb{R}^d$ or as a line in space-time $\mathbb{R}^{d+1}$ (which could be, in particular, space-like and act on the Hilbert space).

Even though the theory away from the impurity is at a fixed point of the Renormalization Group (RG), the impurity may or may not be at a fixed point. Generally we expect that in the infrared the impurity flows to a fixed point. The system with an impurity at a fixed point would preserve a

$$SO(1,2) \subset SO(d+1,2)$$

subgroup of the conformal symmetry. (For defects invariant under transverse rotations, including all the examples appearing below, we also have an unbroken $SO(d)$.) The $SO(1,2) \simeq SL(2,\mathbb{R})$ consists of the conformal transformation that leave the defect at $x^i = 0$, namely translations in time, scaling symmetry, and the special conformal transformation where $b^i = 0$. On the defect worldline $x^i = 0$, these transformations act by $t \to \frac{at+b}{ct+d}$ with $ad - bc = 1$.

We denote defect operators with a hat $\hat{O}_I(t)$, which are distinguished from bulk local operators $O_i(t,x)$. Bulk operators can be expanded in terms of defect operators via the bulk-defect operator product expansion, which takes the schematic form (ignoring transverse spin indices)

$$O_i(t,x) = \sum_I \frac{A_i^I}{|x|^{\Delta_i - \Delta_I}} \hat{O}_I(t) \ .$$

This subject was reviewed recently.[2] The contribution corresponding to the unit operator on the r.h.s., for which we define $A_i^1 \equiv a_i$, is particularly interesting, since it leads to one-point functions for bulk operators

$$\langle O_i(t,x) \rangle = \frac{a_i}{|x|^{\Delta_i}} \ , \tag{1}$$

which are perhaps some of the simplest observables in the presence of a conformal impurity.

If all the $a_i = 0$ (except for the unit operator in the bulk) then the impurity is screened. Indeed, in the absence of an impurity all one-point functions, other than that of the unit operator, vanish. (In fact, line operators with $a_i = 0$ could be either completely screened or topologically nontrivial but screened. We will not dwell on this distinction since the examples discussed below do not admit topologically nontrivial screened line operators.) In that case the bulk-defect operator product expansion becomes simply Taylor's expansion around $x^i = 0$ and the defect operators are simply the original bulk operators and their derivatives. For nontrivial conformal defects, the space of defect operators and their scaling dimensions need to be determined and cannot be simply obtained from the data about the bulk operators.

To close, we would like to discuss a somewhat abstract observable, which will however turn out to be quite useful. Rotating to Euclidean signature, we can consider the expectation value of the circular loop lying in a plane in our $d+1$ Euclidean dimensions (normalized by the vacuum partition function without the circular loop).

As often happens for partition functions in the continuum limit, such an observable is slightly ambiguous due to local counter-terms. In $d > 1$ there is only one counter-term, the cosmological constant term on the worldline, which is essentially the mass of the impurity $\frac{1}{2\pi}\Lambda \int ds$. For a circular loop of radius R this leads to a contribution to the partition function $\log Z \sim \Lambda R$. In $d = 1$ there is another possible counter-term due to the extrinsic curvature $\Lambda' \int \kappa$, with Λ' a purely imaginary dimensionless constant. This counter-term is not a local functional of the data on the curve in $d > 1$. We see that in any d the real part of $(1 - R\frac{d}{dR}) \log Z$ is a scheme independent (finite) observable. In general this is a complicated function of R and as we increase R we probe the RG flow on the impurity. For a conformal line operator, $(1 - R\frac{d}{dR}) \log Z$ is just a constant, which is often denoted by $\log g$, where g represents the "quantum dimension" of the impurity. A simple screened impurity has $\log g = 0$, where "simple" means that it cannot be written as a direct sum of two other impurities.

If the bulk is a topological theory in $d = 2$ then it is well known that $\log g \geq 0$ and the terminology "quantum dimension" originates from the fact that if k impurities are inserted on S^2 then the dimension of the Hilbert space scales like g^k for large k. If the bulk is a conformal theory then we will see that $\log g$ can be either negative or positive. Intuitively, the quantum dimension can be smaller than 1 since the impurity may expel the degrees of freedom around it, which cannot happen in topological theories, since there are no local bulk degrees of freedom to begin with.

Unlike the space of bulk local operators $O_i(t,x)$, about which a lot is known, it is not known how many nontrivial (non-screened, simple[a]) conformal line operators exist even in the most familiar nontrivial fixed points. Several constructions are known for interesting line operators: magnetic defects, spin defects (as in the Kondo problem[3]), and Wilson and 't Hooft lines, which correspond to external electrically or magnetically charged impurities. Whether such defects flow to nontrivial defect fixed points is the subject of the following sections, after we discuss in some detail the subject of RG flows on the impurity.

2. RG Flows

Starting from a conformal defect, which may or may not be trivial, we can trigger an RG flow if we deform the defect action by integrating a defect operator with $\Delta < 1$ (or a marginally relevant defect operator with $\Delta = 1$)

$$\delta S = \mu^{1-\Delta_{\hat{O}}} \int dt \hat{O}(t) \ ,$$

with μ the physical scale in the problem. This drives the theory on the impurity to an infrared fixed (which may be trivial).

Conformal invariance is violated throughout the RG flow, and is only restored at very short or very long distances. Therefore, unlike (1) (which is valid at the defect fixed points), the formula for the one-point function is significantly less constrained

$$\langle O_i(t,x)\rangle = \frac{g(\mu|x|)}{|x|^{\Delta_i}} \ ,$$

with g a function that needs to approach the constants a_i^{uv} and a_i^{ir} for small $|x|$ and large $|x|$, respectively.

A more abstract observable that follows the RG flow is the partition function of the circular loop in Eucildean signature. As we have seen, $(1 - R\frac{d}{dR})\log Z$ is an R independent constant, $\log g$, for a conformal line defect. If there is an RG flow, then it is useful to define a function of R, the defect entropy function

$$s(R) = (1 - R\frac{d}{dR})\log Z \ ,$$

which interpolates between $\log g_{uv}$ and $\log g_{ir}$ for small and large R, respectively.

This quantity obeys the following interesting identity[4]

$$\frac{ds}{dR} = -R^2 \int d\phi_1 d\phi_2 \langle \hat{T}(\phi_1)\hat{T}(\phi_2)\rangle_c (1 - \cos(\phi_1 - \phi_2)) \leq 0 \ .$$

[a]For a non-simple line operator, the Hilbert space is a direct sum, and it is thus sufficient to restrict to simple line operators without loss of generality.

The operator $\hat{T}$ measures the energy density on the defect. It identically vanishes for conformally invariant impurities (the energy cannot be localized on a conformally invariant impurity as it tends to spread to the bulk, but for impurities which are not conformally invariant, the energy can be localized), consistent with that $ds/dR = 0$ at fixed points. This identity shows that in nontrivial RG flows we must have

$$\log g_{uv} > \log g_{ir} \tag{2}$$

and that $\log g$ is constant on conformal manifolds (i.e. it is constant under exactly marginal deformations). Intuitively, it means that as we coarse grain, the number of effective degrees of freedom near the defect decreases (and correspondingly, intuitively, the number of defect operators should decrease as less degrees of freedom remain near the impurity).

Equation (2) shows that the space of conformal impurities is foliated by $\log g$ and that defect RG flows are irreversible. We will see in the next section some applications of this general result. Our discussion is valid in arbitrary d, generalizing the classic results[5–7] in $d = 1$.

3. Impurities in Magnets

In this section we will study some defects in the Wilson-Fisher fixed points[8] in 2+1 dimensions. We will assume that the bulk fields are at the infrared fixed point of the following action[b]

$$S = \int d^2x dt \left(\frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi} - \frac{\lambda_*}{4!} (\vec{\phi} \cdot \vec{\phi})^2 \right) .$$

with $\vec{\phi} = (\phi_1, ..., \phi_N)$. This fixed point is nontrivial for all $N \geq 1$ and has $O(N)$ global symmetry. For what follows, we need to remind of a few of the salient properties of this family of fixed points:

- The dimension of the order parameter $\Delta(\vec{\phi})$ approaches 0.5 from above at large N and its N dependence is rather weak. For $N = 1$, $\Delta(\phi) = 0.518...$.
- The scaling dimension $\Delta(\vec{\phi} \cdot \vec{\phi})$ approaches 2 from below at large N and it again has comparatively weak N dependence and for $N = 1$, $\Delta(\phi^2) = 1.412...$.

3.1. *Pinning Fields*

An example of a pinning field defect is the line operator

$$e^{-h \int dt \phi_1(t, x=0)} . \tag{3}$$

The physical meaning of this line operator is that we have turned on a localized magnetic field in the 1 direction of $O(N)$ at the location $x = 0$. This is a point-like

[b]Of course where the fixed point lies in coupling space is a scheme dependent question which will not concern us here.

impurity corresponding to turning on an external field at a point. Such perturbations are sometimes referred to as pinning field defects (we usually study the response of the system at the phase transition to turning a uniform magnetic field, by contrast, here we turn it on only around a few lattice sites).

The theory with this defect now only preserves $O(N-1)$ symmetry. The ultraviolet fixed point of the defect is at $h=0$, which is a trivial screened defect. Since $\Delta(\phi_1)<1$ for all N, the coupling constant h is relevant at the ultraviolet fixed point $h=0$. The pinning field defect therefore describes an RG flow from the trivial line defect in the ultraviolet to some infrared defect fixed point. Since the ultraviolet $h=0$ fixed point is trivial and the deformation is relevant we immediately learn from (2) that $g_{ir}<1$ which implies that the infrared limit of the defect cannot be trivial! This means that the localized magnetic field is not screened in the infrared, which is a nice application of the monotonicity theorem.

We can think about the line defect as an insertion into the action:

$$S=\int d^2xdt\left(\frac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi}-\frac{\lambda_*}{4!}(\vec{\phi}\cdot\vec{\phi})^2\right)-h\int dt\ \phi_1(t,x=0)\ .$$

As with the Wilson-Fisher fixed point, the infrared limit of the line defect (3) can be understood analytically in the ϵ-expansion, and, in addition, it turns out that an exact large N solution exists. There is some evidence[9] from the ϵ expansion for a mild N dependence for certain observables with the pinning field defect (as with more familiar observables in the Wilson-Fisher fixed point). Therefore let us simply quote results from the large N solution in 2+1 dimensions. The large N solution is obtained by identifying a $SO(1,2)$ invariant saddle point corresponding to the pinning field.[9] It may be possible to solve the entire RG flow for the pinning field parameter h at large N, but at present only results at the infrared (and trivial ultraviolet) fixed point are available. The saddle point at large N is quite complicated (but explicit) and when we quote the predictions below, we retain a few digits after the decimal point. For the infrared fixed point, we find the following g function

$$\log g_{ir}=-0.153673N\ ,$$

which shows that, unlike defects in topological theories, the quantum dimension of conformal defects can be smaller than 1 (and arbitrarily close to 0). More easily accessible quantities are the defect scaling dimensions and expectation values of bulk operators. Consider the bulk order parameter operator $\phi_1(t,x)$ which has scaling dimension 1/2 in the large N limit. Due to the pinning field defect it has a decaying expectation value as in (1). We find from the large N solution

$$\frac{\langle\phi_1(0,x)\rangle}{\sqrt{\langle\phi_1(0,\infty)\phi_1(0,0)\rangle_{\text{no defect}}}}=\frac{a_\phi}{|x|^{1/2}}\ ,\quad a_\phi^2=0.55813N\ . \tag{4}$$

Finally, the order parameter can be brought close to the defect and in addition to the unit operator we have the defect operator $\hat{\phi}_1(t)$, which measures the response of the infrared fixed point to changing the value of the magnetic field away from criticality. We find

$$\Delta(\hat{\phi}_1)=1.542\ . \tag{5}$$

Note that in the ultraviolet $h = 0$ fixed point $\Delta(\hat{\phi}_1) = 0.5$, which shows strong renormalization of the defect operator dimensions due to the RG on the impurity. While we reported here results from the large N limit only, they are in rough agreement with what is seen in simulations at small N.[10]

3.2. *Spin Impurities*

The pinning field defect originates from turning on a localized external field. A more general class of constructions is to allow for new degrees of freedom at the defect. Here we will focus on the $O(3)$ Wilson-Fisher fixed point in $d = 2+1$. While the bulk operators transform under the orthogonal group, we will now add a defect in a $2s+1$ dimensional representation of the $so(3)$ algebra, coupled to the bulk. For half integer spin s, this cannot be completely screened by the bulk degrees of freedom, but it can still flow to a trivial non-simple defect (e.g. a decoupled qubit).

The defect Lagrangian, before coupling to the bulk, is

$$S_{QM} = i \int dt \bar{z}\dot{z} \ , \qquad \bar{z}z = 2s \ .$$

$z = (z_1, z_2)$ are two complex variables on the line subject to the constraint $\bar{z}z = 2s$ and subject to the gauge transformation $z \to e^{i\alpha(t)}z$. The target space of this QM model is therefore S^2 and the Hilbert space consists of $2s+1$ degenerate states in the spin s representation of $so(3)$.[c] The $so(3)$ operators $S^a = \frac{1}{2}\bar{z}\sigma^a z$ can be coupled to the bulk degrees of freedom via

$$S = S_{O(3)} + S_{QM} - \gamma \int dt \sum_{a=1}^{3} S^a \phi_a \ ,$$

where γ is a coupling constant. For $\gamma = 0$ we have a decoupled defect with $2s+1$ states and $\log g_{uv} = \log(2s+1)$. In the infrared we expect to obtain a fixed points, but whether this fixed point is decoupled or not may depend on s. Simulations suggest that there is an infrared fixed point all the way to $s = 1/2$.[11,12] This defect, unlike the pinning field defect, preserves the $SO(N)$ symmetry of the model and hence the one-point function of the order parameter vanishes (in contrast to (4)). However, the one-point function of singlet operators, such as $\vec{\phi}^2$, does not vanish.

At large enough s, there is a nontrivial infrared fixed point. Furthermore, there are several exact results at large s. The main idea is that there is an expansion in $1/s$, where at $s = \infty$ there are reliable saddle points with small fluctuations of S^a.[13] We need to sum over all these saddle points. This allows to obtain several exact results that we quote below. An interesting defect operator is S^a itself. In the ultraviolet fixed point it has dimension 0. But in the nontrivial infrared fixed point, at large s, we find $\Delta(S) \sim 1/s^2$. The coefficient of the $1/s^2$ term is not yet known exactly (in the ϵ expansion one finds $\Delta(S) = \frac{11}{4\pi^2 s^2}$ for sufficiently large

[c] One can think about this QM model as the low-energy approximation of a charged particle moving around a magnetic monopole of charge 2s .

$s \gg \frac{1}{\sqrt{\epsilon}}$). Another interesting defect operator is $\hat{\phi}_a$, for which we know the large s scaling dimension:

$$\Delta(\hat{\phi}_a) = 1 + \mathcal{O}(1/s) \ .$$

Another interesting fact is that the one-point function of $\vec{\phi}^2$ in the large s limit exactly coincides with the one-point function of $\vec{\phi}^2$ in the pinning field defect, up to $\mathcal{O}(1/s)$ corrections. Finally, it is possible to show that in the large s limit the defect entropy at the infrared fixed point is smaller than $\log(2s+1)$ by the (negative) $\log g$ of the $N = 3$ pinning field defect. Here we discussed only spin impurities in the $O(3)$ model, other possible models can be found in.[14]

4. Charged Impurities in Deconfined Critical Points

In 2+1 dimensions there exist fixed points with deconfined (fluctuating) gauge fields. A class of interesting defects are external charged particles (charged impurities). These are described by Wilson lines. We will only make remarks here about 2+1 dimensional QED_3 with $2N_f$ Dirac fermions of charge 1:

$$L = \frac{-1}{4e_*^2} F^2 + i \sum_{a=1}^{2N_f} \bar{\Psi}_a \gamma^\mu (\partial_\mu - iA_\mu) \Psi_a \ .$$

By e_* we refer to the fact that we flow to the infrared fixed point of this bulk deconfined theory. At large N_f this fixed point can be understood in a $1/N_f$ expansion.[15]

Inserting a charged particle of charge q corresponds to deforming the action by $\Delta S = q \int dt A_0(t, x = 0)$. This is the usual charge q Abelian Wilson line. The influence of this charged particle on the bulk can be understood in two steps. First we need to determine the strength of the electric field that this probe causes and then, in the second step, we have to check for self-consistency, i.e. whether the bulk Ψ_a fermions can propagate in a healthy fashion in such an electric field.

To find the electric field we integrate out Ψ_a and obtain (in Euclidean signature)

$$S = -2N_f \mathrm{Tr} \log(\gamma^\mu \partial_\mu - i\gamma^\mu A_\mu) + iq \int d\tau A_0 \ . \tag{6}$$

The saddle point, which is reliable at large N_f, allows to determine the electric field to leading order

$$F_{\tau i} = iE \frac{x^i}{|x|^3} \ , \qquad E = \frac{4q}{\pi N_f} + \mathcal{O}\left(\frac{q^2}{N_f^2}\right) \ . \tag{7}$$

For $q \sim N_f$, at large N_f, the formula for the electric field is more complicated, but it can be obtained by a numerical solution of the saddle point of (6).[16] For concreteness, we provide a plot of $E(q/N_f)$ for large N_f in figure 2.

Now let us ask if the fermions Ψ_a can self-consistently propagate in such a background with an electric field. For this we consider fermion bilinears $\hat{\bar{\Psi}}\hat{\Psi}$ and compute their scaling dimensions as defect operators (in the bulk, at large N_f,

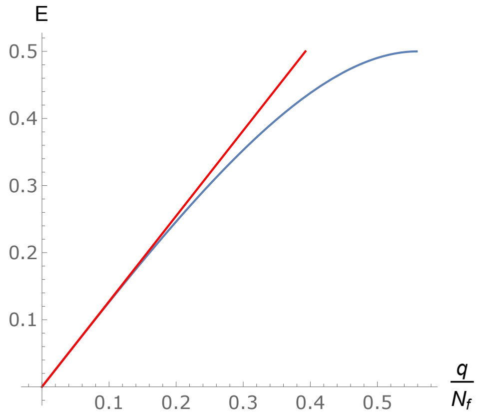

Fig. 2. We plot $E(q/N_f)$ for large N_f. The red line represents the linear approximation $E = \frac{4q}{\pi N_f}$ while the blue line is the correct value of the electric field, taking into account the non-linear nature of the saddle point equation that is obtained from the action (6).

their scaling dimension is close to 2, but there are large corrections to the scaling dimension near the defect). We find[16] that at leading order in N_f

$$\Delta(\hat{\bar{\Psi}}\hat{\Psi}) = 1 + \sqrt{1 - 4E^2} \ .$$

(In the absence of a defect, $E = q = 0$, the scaling dimension of the fermion bilinear of course agrees with the scaling dimension of the bulk primary.) When the scaling dimension turns complex, the fermions are expected to condense and integrating them out naively as we did before (6) is not self-consistent. Therefore we must have $|E| \leq 1/2$ for stability of the vacuum with the Coulomb field.[d] If $|E| \leq 1/2$ then the infrared is a conformal defect with the electric field one-point function (7). If $|E| > 1/2$ then conjecturally there is an RG flow on the defect that reduces q until $|E| = 1/2$. The RG flow on the defect can be understood as a flow of the defect bilinear operator $\hat{\bar{\Psi}}\hat{\Psi}$.

Numerically, from figure 2, we find that $|E| = 1/2$ corresponds to $|q_c| \sim 0.56N_f$ (for large N_f). Therefore Wilson lines with $|q| > |q_c|$ are partially screened and in the infrared flow to the conformal defect with $|q| = |q_c|$. While the Wilson lines with $|q| < |q_c|$ correspond to the conformal defects with $E = E(q, N_f)$ obtained from the the saddle point of (6).

A curious point is that the massive phases of QED_3 are $U(1)_{\pm N_f}$ Chern-Simons theory. (And the critical point is a non-Landau-Ginzburg transition between these topological theories.) These massive phases are obtained by deforming the theory with the $SU(2N_f)$ symmetric mass term for the fermions which can be taken to be either large and positive or large and negative resulting in the topological theories $U(1)_{\pm N_f}$, respectively. Each of these topological theories admits N_f lines with

[d]This phenomenon of vacuum instability with $|E| > 1/2$ was already observed in Graphene.[17]

nontrivial mutual braiding. Since $|q_c| \sim 0.56 N_f$, we have "slightly" more than N_f distinct conformal lines at the fixed point. We therefore have just enough distinct conformal lines to provide us with the nontrivial lines in the topological phases.

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